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Shock Formation in Conical Nozzles

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Evaluation of axially symmetric gas flow in conventional conical nozzles by the method of characteristics has revealed the possibility of shock formation within the nozzle. Negative Mach lines, originating at or just downstream of the junction of the throat profile and the cone, intersect near the axis. Modified computer programs have been run to determine the nature of this phenomenon, and it is concluded that the effect is real. It is shown that the shock formation can be removed by changes in the wall contour near the junction.

Introduction

In a recent paper by Migdal and Landis,[1] the performance of supersonic conical nozzles was investigated by application of the method of characteristics. The authors of this present note began a similar program of work during 1960, but it soon was found that, with the conventional conical nozzle, consisting of a circular-arc contour joined to an expansion cone, intersections of Mach lines of the same family occurred near the axis, resulting in negative flow angles and reduced axial Mach numbers. The first intersection took place along the negative characteristic originating from, or just downstream of, the junction of the two profiles.

Three explanations were put forward for this phenomenon. Firstly, the effect might be a true one, and shock formation could be taking place near the axis; although there are no discontinuities in nozzle slope at the junction, there is a change in the rate of expansion, and this could lead to intersection of Mach lines. In support of this, it was noted that such a situation does not exist with parallel-flow or Rao type nozzles, for which satisfactory results had been obtained using the same basic calculating techniques. Secondly, the accuracy used in the computations might not be sufficient and this could be leading to cumulative errors. Thirdly, the initial conditions in the transonic region might not have been defined sufficiently well.

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Although these last two explanations were considered improbable because of the site of the intersections, it was decided to run several new computer programs to examine these points. The purpose of this paper is to describe the various modifications tried and to report the results, which have led to the conclusion that shock formation occurs. One possible method of removing the shock by a small change in the wall contour is discussed also.

Application of the Method of Characteristics

The basic method of characteristics used to evaluate steady, irrotational, homentropic flows with axial symmetry is described in many text books.[2] The equations of flow yield relations for the variations in flow properties along two families of characteristic (Mach) lines; if conditions at two points not on the same Mach line are known, it is possible to find conditions at a third point by using the relations in finite difference form, together with an iterative procedure. Special relations are involved for points on the axis or for a resultant point on the nozzle wall.

Similar methods were used throughout this work, except that the number of unknowns was reduced by employing the Mach number in place of the velocity. The required flow parameters at any point were thus the Mach number and the angle of the flow to the nozzle axis, together with, of course, the (constant) ratio of specific heats.

The iterative procedure for determining the coordinates and flow properties at the third point is based upon a first approximation, which uses values at the two known points to find coefficients of differentials, followed by further successive approximations using average coefficients. This latter step can be done in two ways, either by using mean coefficients or coefficients of mean values. Either way is suitable for the conventional type of nozzle, but, where large changes in flow direction occur (as in E-D nozzles), the latter process must be employed; consequently, it was applied throughout this work.

For points on the nozzle axis, special iterative relations, which do not appear in the literature, had to be developed. These were obtained by expanding terms involving the ratio of the tangent of the flow angle and the radial distance from the axis in appropriate Taylor series.

In order to begin the numerical integration, conditions have to be prescribed at a sufficient number of points not lying on the same characteristic. This can be accomplished by analyzing the flow in the region of the throat analytically. Sauer[3] has shown how the shape of the sonic line can be determined from small perturbation theory, and his methods can be extended to give curves of constant Mach number in the throat region. Hall, in a more recent publication,[4] derived expressions from which second- and third-order corrections also can be evaluated.

The methods used have been based mainly on Sauer's first-order theory, although Hall's relations have been applied also. However, the curve of constant Mach number joining the points on the nozzle wall at the throat section could not be used, since opposite characteristics from some of the points on this curve intersect upstream of the curve. To avoid this, conditions were derived along straight lines, joining the points on the nozzle wall at the throat section and the intersection of the constant Mach number parabola through these points with the axis of symmetry. The Mach number and flow direction along these lines then could be computed easily.

Except for cases mentioned in the next section, the following calculating procedure was adopted. Initial conditions were determined at twenty-one equally spaced points between the axis and the throat wall, the values being calculated and stored to the full machine capacity. The pattern used was to calculate along negative characteristics, beginning with the point on the nozzle wall at the throat section. However, after initial trials, negative characteristics from the other starting points were not used, since these led to a congested band of Mach lines and the possibility of errors being intro-

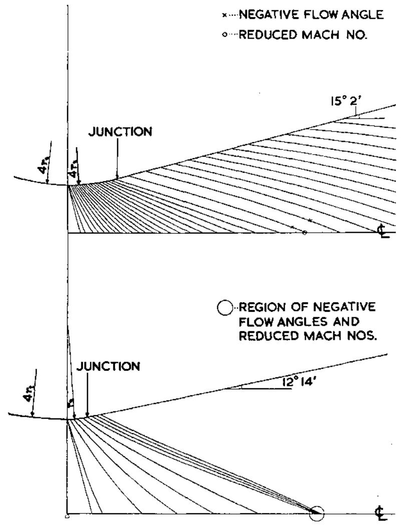

Fig. 1 Pattern of negative characteristics for two conventional conical nozzles.

duced near the axis. The iterative process was continued until the coordinates and flow parameters at the point being determined were such that the absolute difference between successive approximations was less than or equal to 1×10^{-5} of the final value. The work was programed on a Mercury digital computer.

Results

Programs have been run for cone semiangles of 12°, 15°, and 25° and for radii of curvature at the throat section (compared to the throat radius) of 2.0 and 4.0 upstream and 0.4, 1.0, 4.0 downstream. The ratio of specific heats was taken as 1.24.

In all cases negative flow angles were encountered just off the axis on either the first or second negative characteristic originating downstream of the circular-arc and conical profiles. The Mach number on the axis also was reduced substantially when this occurred. In some instances, the program continued to run and satisfactory flow values were obtained along succeeding characteristics, but, in others, the calculations broke down due to the appearance of subsonic Mach numbers. Figure 1 shows the negative characteristics for two typical cases.

In order to test the hypothesis that successive approximations might be leading to cumulative errors, a program was repeated with the basic accuracy changed from 1×10^{-5} to 5×10^{-6}. The two sets of results were virtually identical.

The effects of conditions in the transonic region were investigated in two ways, firstly by changing the number of initial points and secondly by using the higher-order approximations due to Hall.

Increasing the initial points from 21 to 41 resulted in only negligible differences. A decrease in number from 21 to 11, although still resulting in negative flow angles and reduced Mach numbers, had a significant effect on the accuracy. The third-order approximations in the throat region brought about only minor changes.

The mesh sizes resulting from the calculations depend upon the number and position of the initial points. For equally spaced starting points, reflected characteristics give rise to some large mesh sizes further downstream. To see if this could result in inaccuracies, several runs were made with an increased number of initial points, which were spaced irregularly so as to result in a more even mesh. These again gave negative flow angles and reduced Mach numbers, the detailed results being very similar to those from the original program.

The iterative procedure used near the nozzle axis was examined by a trial in which the basic finite difference equa-

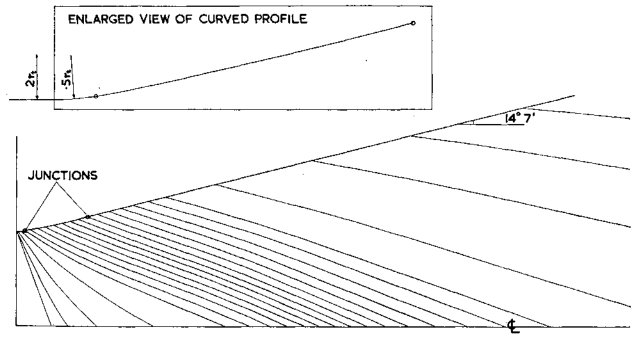

Fig. 2 Pattern of negative characteristics for parallel flow-conical nozzle.

tions were used for all points (this is possible if coefficients are evaluated from mean values). Again only slight changes were observed.

The conclusion then was reached that shock formation was occurring near the axis, due to the geometrical shape of the circular arc-conical nozzle. Modifications to the curved profile were proposed to overcome this. The first attempt used a "quartic" shape, which is somewhat similar to the circular arc but has a continuous curvature as well as slope at the junction with the cone. Again negative flow angles and reduced axial Mach numbers were encountered; by careful choice of the initial points it was possible to show that this first happened along the negative characteristic originating from the junction. With a circular-arc profile it had been shown, in a similar manner, that these conditions first arose just downstream of the junction characteristic and it thus appeared that the only effect of the quartic profile was to move the shock slightly upstream.

The second attempt was based on the contours developed for parallel-flow nozzles. In these calculations a circular-arc contour is used downstream of the throat and is terminated at a point from which the negative characteristic gives the required exit Mach number on the axis. The wall shape then is determined in the usual manner; the slope continues to increase from the end of the circular section, reaching a maximum at a point of inflection further downstream. A conical portion, fitted at this point of inflection, will lie outside the parallel-flow contour and thus will result in widening of negative characteristics (conversely, if fitted at the end of a circular arc, some compression can be expected to occur). A program has been run for which a parallel-flow entry shape joined to a cone was used, the wall angle at the point of inflection being approximately 14°. The results are illustrated in Fig. 2, and the curves of constant Mach number (and pressure) are shown in Fig. 3. Satisfactory values were obtained throughout for the flow angles and Mach numbers, and it can be seen that there was no tendency for negative characteristics to intersect near either of the two junctions.

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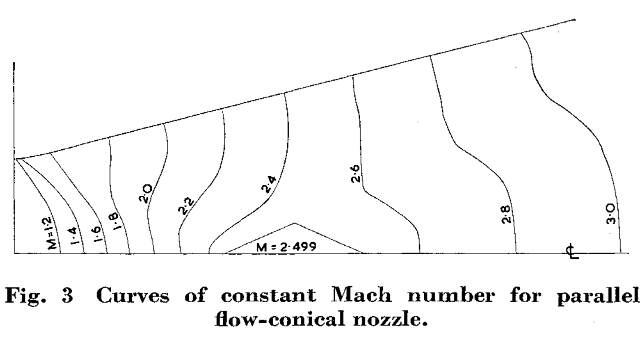

Fig. 3 Curves of constant Mach number for parallel flow-conical nozzle.

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Prediction of Chamber Pressure Decay Transients during Termination of Solid Propellant Rocket Motors

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Nomenclature

a = reaction constant evaluated from experimental data
A_b = total burning surface area of propellant
A_n = combined area of nozzle throats
A_t = combined area of termination ports
b = diffusion constant evaluated from experimental data
c = constant evaluated from experimental data
C_d = constriction coefficient for orifice-type ports
C_w = mass flow coefficient for choked isentropic flow, i.e.,

$$C_w = \left[\frac{kg}{RT}\left(\frac{2}{k+1}\right)^{(k+1)/(k-1)}\right]^{1/2}$$

M = molecular weight of combustion products
n = constant, evaluated from experimental data
P = chamber pressure at time t
ΔP = 50% reduction from initial chamber pressure, i.e., $\Delta P = \frac{1}{2}P_0$
r = propellant burning rate
R = universal gas constant
t = time after thrust termination
Δt = time interval for 50% reduction from initial chamber pressure
T = stagnation temperature of chamber
V = chamber void volume
w = total mass of gaseous combustion products in chamber
ρ_p = propellant density

Subscript

0 = value at initial termination, i.e., when $t = 0$

Introduction

THE prediction of rocket motor performance during the unsteady-state periods of ignition and termination necessitates an accurate knowledge of the effects of chamber pressure on propellant burning rate. Vieille's law and Summerfield's equation are two such means of relating propellant burning rate to chamber pressure. This note reports an investigation in which Summerfield's equation was found to be more accurate than Vieille's law in predicting the effect of chamber pressure on the burning rate of a composite, double-base, solid propellant rocket motor. This paper also discusses the combination of a mass balance with Summerfield's equation, resulting in an accurate means of predicting pressure decay transients when a solid propellant rocket motor is terminated by opening auxiliary ports in the chamber sides.

Comparison of Vieille's Law to Summerfield's Equation for a Double-Base Propellant

One often used relationship for predicting the effect of chamber pressure on propellant burning rates is Vieille's law, which states that

$$r = cP^n \quad (1)$$

More recently, Summerfield et al.[1] developed a burning rate equation from diffusional mixing and chemical reaction theory. The equation is intended for use with a heterogeneous solid propellant where the crystals and matrix are assumed to burn at the same rate. Summerfield's equation does not allow prediction of burning rate directly but does predict the pressure dependence of the burning rate to be

$$1/r = (a/P) + (b/P^{1/3}) \quad (2)$$

Summerfield et al. have shown that this equation accurately represents the effect of pressure on the burning rate of selected composite propellant strands for pressures from 14.7 to 1500 psia.

At Hercules Powder Company, a comparison was made between Vieille's law and Summerfield's equation for an aluminized double-base solid propellant for the pressure range of 3 to 1000 psi. It was found that Summerfield's equation more accurately predicted the pressure dependence of the propellant burning rate for this case. Predictions obtained from the two burning rate equations were compared to experimental data accumulated from 65 burning rate experiments with an aluminized double-base solid propellant. These data extended over the pressure range of 3 to 1000 psia and included burning rates from the firing of propellant strands, cylinders, and six different solid propellant rocket motors. Descriptions of some of these motors can be found in Refs. 2 and 3. Vieille's equation correlated with the experimentally determined data for pressures in excess of 200 psi but was unsatisfactory over the entire 3 to 1000 psi range. The burning rates predicted by Summerfield's equation correlated with the experimentally determined burning rates for the full range of chamber pressures. The accuracy of the Summerfield equation is especially notable when considering the wide variety of strand, cylinder, and rocket motor sizes from which the data were obtained. The classified literature[4] contains a detailed presentation and discussion of these data and correlations.

Where rocket chamber pressures vary widely (for example, during ignition and termination), the use of the Summerfield correlation to predict pressure dependence of the burning rate of solid metalized composite or double-base propellants seems to provide the most accurate results. Large errors in prediction of burning rates over wide pressure ranges could

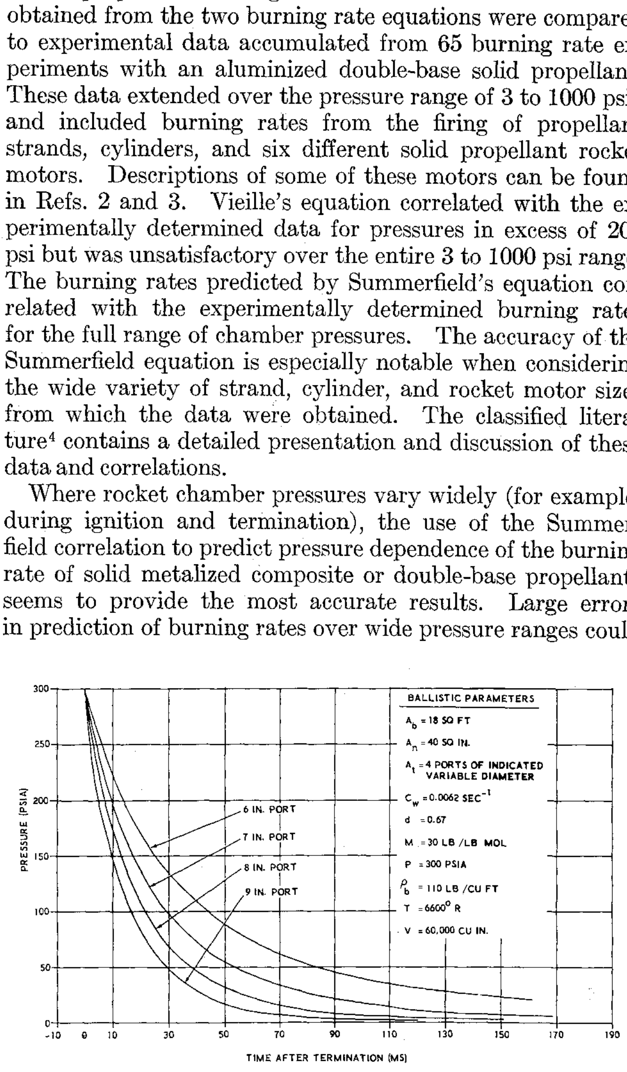

Fig. 1 Predicted effect of port diameter on chamber pressure following thrust termination.

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